Supercuspidal Representations and Symmetric Spaces

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Outline

1. Overview of results with Fiona Murnaghan
   - Harmonic analysis background
   - Distinguished tame supercuspidal representations
   - Equivalence of tame supercuspidal representations
2. Explicit statements
   - The equivalence problem
   - The main theorem
3. Symmetry conditions on data
   - A conjectural symmetry condition
   - Evidence for the conjecture

What is harmonic analysis?

Some popular answers:

- It's the decomposition of $L^2(G)$ into irreducible unitary representations of G, where G is a locally compact group like $SO(2)$ (the circle).
- It's the decomposition of $L^2(H\backslash G)$ into irreducible unitary representations of G, where $H\backslash G$ is a symmetric space like $SO(2)\backslash SO(3)$ (the sphere).

- The group G acts on functions on G by right translations: $(g \cdot f)(x) = f(xg)$.
- The representations that "occur" in the decomposition of $L^2(H\backslash G)$ are called *H*-distinguished representations of G.

The groups of interest to us

- $\mathbf{G}$ will be a connected reductive group defined over a finite extension F of $\mathbb{Q}_p$, for some (finite) odd prime p.
- $G = \mathbf{G}(F)$
- $\mathbf{H} = \mathbf{G}^\theta$ is the group of fixed points of an F-automorphism θ of $\mathbf{G}$ of order two.
- $H = G^\theta = \mathbf{H}(F)$.

Induced representations

We are interested in harmonic analysis on $H\backslash G$.

- But instead of working with $L^2(H\backslash G)$, we work with $C^\infty(H\backslash G)$.
- There are various notions of "induced representation," with each suited to different examples. Both $L^2(H\backslash G)$ and $C^\infty(H\backslash G)$ can be viewed as "the" representation of G induced from the trivial representation of H.
- For some applications, such as the Plancherel formula, it is more appropriate to work with $L^2(H\backslash G)$.

H-distinguished representations

Let (π, V) be an irreducible admissible representation of G.

Frobenius Reciprocity:

- $\mathrm{Hom}_G(\pi, C^\infty(H\backslash G)) \cong \mathrm{Hom}_H(\pi, 1)$

- If either side of Frobenius Reciprocity is nonzero, we say π is *H*-distinguished.
- The elements of $\mathrm{Hom}_H(\pi, 1)$ are just linear forms $\lambda : V \to \mathbb{C}$ such that $\lambda(\pi(h)v) = \lambda(v)$, for all $h \in H$ and $v \in V$.

A provocative example: Quadratic base change for GL_n

- Let E be a quadratic extension of F.
- Let $G = GL_n(E)$.
- If $g \in G$, let $\bar{g} \in G$ be obtained by applying the nontrivial Galois automorphism of E/F to matrix entries.
- Let $\eta \in G$ be "hermitian" (i.e., ${}^t\bar{\eta} = \eta$).
- Let $\theta(g) = \eta\, {}^t\bar{g}^{-1}\eta^{-1}$.
- Then $H = G^\theta$ is a unitary group in n variables.
- Let $\theta'(g) = \bar{g}$ and $H' = G^{\theta'} = GL_n(F)$.

Quadratic base change for GL_n

Let π be an irreducible supercuspidal representation of G.

The following are equivalent:

- π is H-distinguished.
- $\mathrm{Hom}_H(\pi, 1)$ has dimension one.
- π is a base change lift from H'.
- $\pi \simeq \pi \circ \theta'$.
- $L_{\mathrm{Asai}}(s, \pi)$ has a pole at $s = 1/2$.

How typical is this for general G and θ?

$H = G^\theta$

Questions

Suppose we are given G and θ and let π be an irreducible tame supercuspidal representation of G.

Questions:

- (Multiplicity One Question) If π is H-distinguished does $\mathrm{Hom}_H(\pi, 1)$ have dimension one?
- (Liftings) Can the set of distinguished representations be described as the image of a "lifting"?
- (Symmetry conditions) Do the distinguished representations satisfy a symmetry condition?
- (Special values of *L*-functions) Is there a correlation between distinguishedness (a.k.a, distinction) and an *L*-function condition?

Reducibility of induced representations

A given G may embed as a Levi factor in a larger group $\widetilde{G}$. Then one can consider when a representation of G yields an irreducible representation of $\widetilde{G}$ via parabolic induction.

Murnaghan and Repka studied this (following a general approach of Shahidi). In some cases, one can find θ so that a representation of G induces an irreducible representations of $\widetilde{G}$ exactly when it is G^θ-distinguished.

In other cases, it is the opposite (distinguishedness correlates with reducibility of induced representations).

The representations of interest to us

Fiona Murnaghan (U of Toronto) and I studied and classified (to some extent) the tame supercuspidal H-distinguished representations of G.

- The tame supercuspidal representations of G are precisely the supercuspidal representations constructed by Jiu-Kang Yu.
- Yu defines a certain set of parameters and associates to each parameter Ψ an irreducible supercuspidal representation $\pi(\Psi)$.

The flavor of the main result

Under mild conditions, we compute $\mathrm{Hom}_H(\pi(\Psi), 1)$ (sort of).

- We actually show that $\mathrm{Hom}_H(\pi(\Psi), 1)$ is equivalent to an analogous object involving "depth zero" things.
- In principle, this might be a reduction from p-adic harmonic analysis to harmonic analysis over finite fields.
- Our results allow one to detect the distinguishedness (a.k.a., distinction) of $\pi(\Psi)$ from properties of the Yu datum Ψ.

An unintended consequence of the main result

The equivalence question:
When do two Yu data Ψ_1 and Ψ_2 yield equivalent representations $\pi(\Psi_1)$ and $\pi(\Psi_2)$?

- This was a question that resisted straightforward attempts at a solution.
- It turns out that the solution follows from our general study of the spaces of invariant linear forms mentioned earlier.

A companion question

The exhaustion question:
Does Yu's construction "almost always" yield all of the irreducible supercuspidal representations of G?

- Ju-Lee Kim established an affirmative answer to this question.
- So our result describes the fibers of $\Psi \mapsto \pi(\Psi)$ and Kim's result determines the image.

The connection between the main result and the equivalence question

- STEP 1: (The contragredient of a datum)
  Given a G-datum Ψ, we define a new datum $\widetilde{\Psi}$ such that $\pi(\widetilde{\Psi})$ is contragredient to $\pi(\Psi)$. Explicitly:
  $$(\vec{\mathbf{G}}, y, \rho, \vec{\phi})^\sim = (\vec{\mathbf{G}}, y, \tilde{\rho}, \vec{\phi}^{-1}).$$
- STEP 2: (The product of data)
  Given a G_1-datum Ψ_1 and a G_2-datum Ψ_2, we define a $(G_1 \times G_2)$-datum $\Psi_1 \times \Psi_2$ such that $\pi(\Psi_1 \times \Psi_2) \simeq \pi(\Psi_1) \times \pi(\Psi_2)$.

The connection between the main result and the equivalence question (continued)

View $G\backslash(G \times G)$ as a symmetric space via the involution $\theta(g_1, g_2) = (g_2, g_1)$. Given G-data Ψ_1 and Ψ_2, it is easy to see that $\pi(\Psi_1) \simeq \pi(\Psi_2)$ if and only if $\pi(\Psi_1) \times \pi(\Psi_2)^\sim$ is G-distinguished.

The connection:
Ψ_1 and Ψ_2 yield equivalent representations of G, exactly when $\Psi_1 \times \widetilde{\Psi}_2$ yields a G-distinguished representation of $G \times G$.

- The main theorem tells when $\pi(\Psi_1 \times \widetilde{\Psi}_2)$ is distinguished.

What is a Yu datum?

A Yu datum (a.k.a., a cuspidal G-datum) is a 4-tuple (a.k.a., a quadruple): $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$, where

- $\vec{\mathbf{G}} = (\mathbf{G}^0, \dots, \mathbf{G}^d)$ is a tower $\mathbf{G}^0 \subset \cdots \subset \mathbf{G}^d = \mathbf{G}$ of subgroups (called a twisted Levi sequence).
- y is a point in the Bruhat-Tits building of G.
- ρ is a depth zero representation of $G^0_{[y]}$, the isotropy group of the image $[y]$ of y in the reduced building.
- $\vec{\phi} = (\phi_0, \dots, \phi_d)$, where ϕ_i is a quasicharacter of $G^i = \mathbf{G}^i(F)$.

Ways to alter Ψ without changing $\pi(\Psi)$

There is a way to alter $\vec{\phi}$ without affecting
$$\prod_{i=0}^{d} \phi_i |(\text{a suitable subgroup of } G^0).$$

This type of manipulation is called a refactorization.

- Refactoring doesn't affect $\pi(\Psi)$.
- To be honest, ρ is also involved in the definition of "refactorization."
- Refactorizations are directly related to Roger Howe's refactorizations of the "Howe factorizations" that occur in his GL_n construction.

Ways to alter Ψ without changing $\pi(\Psi)$ (continued)

If α is any F-automorphism of **G** and Ψ is a G-datum then there is a natural way to obtain a new G-datum Ψ^α.

If $\alpha = \mathrm{Int}(g)$ is conjugation by an element of $g \in G$ then we write $g \cdot \Psi = \Psi^\alpha$.

- $\pi(g \cdot \Psi) \simeq \pi(\Psi)$.
- We say $g \cdot \Psi$ and Ψ are G-conjugate.

Explicit answer to the equivalence question

The equivalence answer:
Two Yu data Ψ_1 and Ψ_2 yield equivalent representations $\pi(\Psi_1)$ and $\pi(\Psi_2)$ exactly when they are related by a combination of refactorization and G-conjugation.

- As usual, I am lying.
- One is also allowed to replace ρ by an equivalent representation.
- One can also replace y by another point with the same ~~image in the reduced building.~~

The Main Theorem

To the left is an artistic rendering of the Main Theorem. I will try to explain the symbols in stages.

The top level formula

$$\langle \Theta, \xi \rangle_G = m_K(\Theta) \sum_{\Theta' \in \Theta^K} \langle \Theta', \xi \rangle_K$$

- The left side of the formula is essentially the dimension of a space $\mathrm{Hom}_{G^\theta}(\pi(\Psi), 1)$.
- Let me explain in more detail ...

The Main Theorem

The top level formula

$$\langle \Theta, \xi \rangle_G = m_K(\Theta) \sum_{\Theta' \in \Theta^K} \langle \Theta', \xi \rangle_K$$

- Suppose Ψ is a Yu datum.
- Associated to Ψ is an open, compact-mod-center subgroup $K(\Psi)$ of G and an irreducible representation $\kappa(\Psi)$ of $K(\Psi)$.
- The representation $\pi(\Psi)$ is the representation c-$\mathrm{Ind}_{K(\Psi)}^G(\kappa(\Psi))$ obtained by compactly-supported induction from $\kappa(\Psi)$.

The Main Theorem

The top level formula

$$\langle \Theta, \xi \rangle_G = m_K(\Theta) \sum_{\Theta' \in \Theta^K} \langle \Theta', \xi \rangle_K$$

- In the above formula, K is any subgroup of G of the form $K(\Psi)$, for at least one Ψ.
- If Ψ is a G-datum for which $K(\Psi) = K$, we say Ψ is a (G, K)-datum.

The Main Theorem

The top level formula

$$\langle \Theta, \xi \rangle_G = m_K(\Theta) \sum_{\Theta' \in \Theta^K} \langle \Theta', \xi \rangle_K$$

- ξ is an equivalence class of (G, K)-data, where equivalence is defined via refactorizations and *K-conjugation* (not G-conjugation).
- Note: Two (G, K)-data Ψ_1 and Ψ_2 are equivalent exactly when $\kappa(\Psi_1) \simeq \kappa(\Psi_2)$.

The Main Theorem

The top level formula

$$\langle \Theta, \xi \rangle_G = m_K(\Theta) \sum_{\Theta' \in \Theta^K} \langle \Theta', \xi \rangle_K$$

- An involution of G is an F-automorphism of **G** of order two.
- G acts on the set of involutions by $g \cdot \theta = \mathrm{Int}(g) \circ \theta \circ \mathrm{Int}(g)^{-1}$.
- Θ is a G-orbit of involutions.
- Θ^K is the set of K-orbits in Θ.

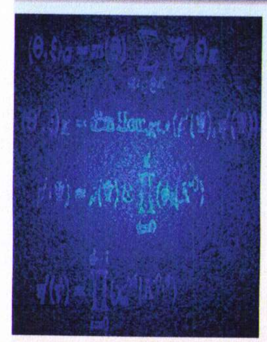
The Main Theorem

The top level formula

$$\langle \Theta, \xi \rangle_G = m_K(\Theta) \sum_{\Theta' \in \Theta^K} \langle \Theta', \xi \rangle_K$$

- $\langle \Theta, \xi \rangle_G = \dim \mathrm{Hom}_{G^\theta}(\pi(\Psi), 1)$, for *any* $\theta \in \Theta$ and *any* $\Psi \in \xi$.
- $\langle \Theta', \xi \rangle_K = \dim \mathrm{Hom}_{K^\theta}(\kappa(\Psi), 1)$, for *any* $\theta \in \Theta'$ and *any* $\Psi \in \xi$.
- $m_K(\Theta)$ is the (finite) number of elements in each fiber of the map $K\backslash G/G^\theta \to \Theta^K$ given by $g \mapsto g \cdot \theta$, for any $\theta \in \Theta$.

The Main Theorem



The auxiliary formulas

$$\langle \Theta', \xi \rangle_K = \dim \mathrm{Hom}_{K^{0,\theta}}(\rho'(\Psi), \eta'(\Psi))$$
$$\rho'(\Psi) = \rho(\Psi) \otimes \prod_{i=0}^{d} \left(\phi_i | K^0\right)$$
$$\eta'(\Psi) = \prod_{i=0}^{d-1} \left(\chi_i^{\mathcal{M}} \mid K^{0,\theta}\right)$$

- This stuff involves a depth zero representation ρ, quasicharacters ϕ_i and a quadratic character η'.

Recall the quadratic base change setup:

- $G = GL_n(E)$, with E/F quadratic.
- $\theta(g) = \eta {}^t\bar{g}^{-1}\eta^{-1}$, where ${}^t\bar{\eta} = \eta$, and $\theta'(g) = \bar{g}$.
- $H = G^\theta = U_n(\eta, E/F)$, $H' = G^{\theta'} = GL_n(F)$.

The following are equivalent:

- π is H-distinguished.
- $\mathrm{Hom}_H(\pi, 1)$ has dimension one.
- π is a base change lift from H'.
- $\pi \simeq \pi \circ \theta'$.

Restatement of the symmetry condition:

Let Ψ be a G-datum in the G-equivalence class ξ.

- The condition $\pi(\Psi) \simeq \pi(\Psi) \circ \theta'$ is the same as $\xi = \theta'(\xi)$.
- The latter condition only depends on the G-orbit Θ' of θ', since $g \cdot \theta' = \mathrm{Int}(g) \circ \theta' \circ \mathrm{Int}(g)^{-1}$ and inner automorphisms preserve G-equivalence classes of data.
- So we write the symmetry condition as

$$\Theta'(\xi) = \xi.$$

Another restatement:

- A theorem of Gelfand-Kazhdan says that if $\tau(g) = {}^t g^{-1}$ then $\tilde{\pi} \simeq \pi \circ \tau$.
- Note that $\tau \circ \theta' = \mathrm{Int}(\eta)^{-1} \circ \theta$.
- If Θ is the G-orbit of θ then $\Theta(\xi) = \tau(\Theta'(\xi)) = \tau(\xi) = \tilde{\xi}$.
- So our symmetry condition can be rewritten as

Symmetry condition:

$\Theta(\xi) = \tilde{\xi}$

Symmetry versus Distinction:

What is the connection between the following conditions?

1. $\Theta(\xi) = \tilde{\xi}$
2. $\langle \Theta, \xi \rangle_G \neq 0$

Conjecture

Condition 2 implies Condition 1.

Self-contragredient representations:

Let $\mathbf{G} = GL_{2n}$ with $\theta = \mathrm{Int}(1_n \oplus (-1_n))$.

- Then $\Theta(\xi) = \xi$.
- So Condition 1 becomes $\xi = \tilde{\xi}$.
- In other words, we are dealing with self-contragredient representations and liftings from classical groups.
- In this case, Condition 1 does not imply Condition 2.

Abelian groups: a heuristic

- Let α be an automorphism of order two of an abelian group A.
- Let $\chi : A \to \mathbb{C}^\times$ be an irreducible representation of A.
- The analogue of Condition 2 for χ and α is the condition that χ is trivial on the group A^α of fixed points of α.
- This implies χ is trivial on the elements of the form $a\alpha(a)$, with $a \in A$.
- This implies $\chi \circ \alpha = \chi^{-1}$, which is the analogue of Condition 1.

θ-symmetry

Proposition

Suppose that in the formula

$$\langle \Theta, \xi \rangle_G = m_K(\Theta) \sum_{\Theta' \in \Theta^K} \langle \Theta', \xi \rangle_K$$

some summand $\langle \Theta', \xi \rangle_K$ is nonzero. Then for every $\theta \in \Theta'$ there exists $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi}) \in \xi$ such that:

- $\theta(\vec{\mathbf{G}}) = \vec{\mathbf{G}}$,
- $\theta[y] = [y]$,
- $\vec{\phi} \circ \theta = \vec{\phi}^{-1}$.
- (No condition on ρ.)

If we only had the additional condition $\rho \circ \theta \simeq \tilde{\rho}$ then we would be able to conclude that $\langle \Theta, \xi \rangle_G \neq 0$ implies that $\Theta(\xi) = \tilde{\xi}$.

Now ρ is nearly a distinguished representation of a cuspidal representation of a finite group of Lie type. It appears that our conjecture may perhaps follow fairly easily from an analogous conjecture over finite fields.

This is the subject of a research project Ryan Vinroot and I have tentatively agreed to pursue.